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Number theory

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Problem Set #4

Recall first the two following result:

Lemma: Let $0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N \longrightarrow 0$ be a short exact sequence; then M is Noetherian (Artinian) if and only if N and L are both Noetherian (Artinian).
Proof: ($\Rightarrow$) *Suppose M is Noetherian. Since L can be viewed as a submodule of M, every submodule of L is a submodule of M which is f.g., therefore L is Noetherian. On the other hand, if N' is a submodule of N, then the submodule $g^{-1}(N')$ is f.g., so that N' is f.g. as g is surjective. Therefore N is also Noetherian.*
($\Leftarrow$) *Let $M_1 \subseteq M_2 \subseteq M_3 \subseteq ...$ be an ascending chain of submodules of M. Since N and L are Noetherian, there is an integer n such that $f^{-1}(M_i) = f^{-1}(M_n)$ and $g(M_i) = g(M_n)$ for every $i \geq n$. To finish the proof, it suffices to show that $M_{n+1} = M_n$. For this, let $x \in M_{n+1}$; then there is an element $y \in M_n$ such that $g(x) = g(y)$, so that $x - y \in ker(g) = Im(f)$, it follows that there is an element $z \in L$ such that $f(z) = x - y$. However, $x - y \in M_{n+1}$ and $z \in L$, $z \in f^{-1}(M_{n+1}) = f^{-1}(M_n)$. Therefore $x - y \in M_n$ as f is 1.1, Hence, $x \in M_n$. The same kind of proof can be written for the Artinian part.*

Lemma: Consider a finite chain of submodule of M, say $(0) = M_0 \subset M_1 \subset M_2 \subset ... \subset M_s = M$. Then M is Artinian (resp. Noetherian) if and only if each quotient module M_{j+1}/M_j is Artinian (resp. Noetherian).
Proof: *Apply the previous lemma inductively for $j \geq 0$ to the sequences*

$$0 \longrightarrow M_j \longrightarrow M_{j+1} \longrightarrow M_{j+1}/M_j \longrightarrow 0$$

Exercise 7 p 23 of [N]
In a noetherian ring R in which every prime ideal is maximal, each descending chain

$$.... \subseteq \mathfrak{a}_2 \subseteq \mathfrak{a}_1$$

becomes stationary, that is R is Artinian.
Solution:
Let R be a noetherian ring in which every prime ideal is maximal.
We ***Claim:*** *Any ideals $\mathfrak{a}$ in R contains a product of prime ideals $\mathfrak{p}_1$, $\mathfrak{p}_2$, ..., $\mathfrak{p}_r$, that is*

$$\mathfrak{p}_1 .... \mathfrak{p}_r \subseteq \mathfrak{a}$$

Indeed, suppose the set $\mathfrak{M}$ of those ideals which do not fulfill this condition is not empty. As R is noetherian, every ascending chain of ideals becomes stationary. Therefore $\mathfrak{M}$

is inductively ordered with respect to inclusion and thus admits a maximal element $\mathfrak{a}$. *This cannot be a prime ideal, so there exist elements* b_1, $b_2 \in R$ *such that* $b_1 b_2 \in \mathfrak{a}$ *but* $b_1, b_2 \notin \mathfrak{a}$. *Put* $\mathfrak{a}_1 = (b_1) + \mathfrak{a}$, $\mathfrak{a}_2 = (b_2) + \mathfrak{a}$. *Then* $\mathfrak{a} \subsetneq \mathfrak{a}_1$, $\mathfrak{a} \subsetneq \mathfrak{a}_2$ *and* $\mathfrak{a}_1\mathfrak{a}_2 \subsetneq \mathfrak{a}$. *By maximality of* $\mathfrak{a}$, *both* $\mathfrak{a}_1$ *and* $\mathfrak{a}_2$ *contain a product of prime ideals, and the product of these product is contained in* $\mathfrak{a}$, *a contradiction.*

In particular, (0) *is a product of prime but since we are assuming that any prime ideal is maximal. We can write* $(0) = \mathfrak{m}_1 ... \mathfrak{m}_n$ *where* $\mathfrak{m}_i$ *are maximal ideals. Now, we prove that this implies the required result. For this consider the sequence*

$$(0) = \mathfrak{m}_1\mathfrak{m}_2...\mathfrak{m}_n \subseteq ... \subseteq \mathfrak{m}_1\mathfrak{m}_2 \subseteq \mathfrak{m}_1 \subseteq R$$

of ideals of R.
Each factors $\mathfrak{m}_1...\mathfrak{m}_{i-1}/\mathfrak{m}_1....\mathfrak{m}_i$ *is a vector space over the field* $R/\mathfrak{m}_i$. *Hence, being Artinian is equivalent of being noetherian for each factor. Now applying the previous lemmas, we obtain that* R *is noetherian if and only if* R *is Artinian.*

Exercise 8 p 23 of [N]
Let $\mathfrak{m}$ be a zero integral ideal of the Dedekind domain $\mathcal{O}$. Show that in every ideal class of Cl_K, there exists an integral ideal prime to $\mathfrak{m}$.

Solution:
Let $\mathfrak{m}$ *be a zero integral ideal of the Dedekind domain* $\mathcal{O}$, *we write*

$$\mathfrak{m} = \mathfrak{p}_1^{v_{\mathfrak{p}_1}(\mathfrak{m})} ... \mathfrak{p}_r^{v_{\mathfrak{p}_r}(\mathfrak{m})}$$

. *We want an integral ideal* $\mathfrak{a}$ *prime to* $\mathfrak{m}$, *i.e. with no common prime ideal in their decomposition such that* $\mathfrak{a}\mathfrak{m}^{-1}$ *is principal let's say equals* (x) *for some* $x \in \mathcal{O}$.

Now, via Chinese remainder theorem as we have already done for exercise 5, we can choose x *so that* $v_{\mathfrak{p}_i}(x) = v_{\mathfrak{p}_i}(\mathfrak{m})$ *for any* i. *Now, put* $\mathfrak{a} = (x)\mathfrak{m}^{-1}$ *then* $v_{\mathfrak{p}_i}(\mathfrak{a}) = 0$ *for any* i *and* $v_{\mathfrak{q}}(\mathfrak{a}) \geq 0$ *so that* $\mathfrak{a}$ *is an integral ideal prime to* $\mathfrak{m}$ *and* $\mathfrak{a}\mathfrak{m}^{-1} = (x)$.

Exercise 9 p 23 of [N]
Let $\mathcal{O}$ be an integral domain in which all nonzero ideals admit a unique factorization into prime ideals. Show that $\mathcal{O}$ is a Dedekind domain.

Solution:
Let $\mathcal{O}$ be an integral domain in which all nonzero ideals admit a unique factorization into prime ideals.
We have that any fractional ideal is invertible and for any fractional ideal $\mathfrak{a}$ its inverse is

$$\mathfrak{a}^{-1} = \{x \in K | x\mathfrak{a} \subseteq \mathcal{O}\}$$

We recall that for any prime ideal $\mathfrak{p}$, $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}$. Indeed $\mathfrak{p} \subsetneq \mathfrak{p}\mathfrak{p}^{-1} \subseteq \mathcal{O}$. Since $\mathfrak{p}$ is maximal, it follows that $\mathfrak{p}\mathfrak{p}^{-1} = \mathcal{O}$. Moreover, if $\mathfrak{a}$ an ideal of $\mathcal{O}$ and

$$\mathfrak{a} = \mathfrak{p}_1^{v_{\mathfrak{p}_1}(\mathfrak{a})} ... \mathfrak{p}_r^{v_{\mathfrak{p}_r}(\mathfrak{a})}$$

Then

$$\mathfrak{b} = \mathfrak{p}_1^{-v_{\mathfrak{p}_1}(\mathfrak{a})} ... \mathfrak{p}_r^{-v_{\mathfrak{p}_r}(\mathfrak{a})}$$

in such that $\mathfrak{b}\mathfrak{a} = \mathcal{O}$, so that $\mathfrak{b} \subseteq \mathfrak{a}^{-1}$. Conversely, if $x\mathfrak{a} \subseteq \mathcal{O}$ then $x\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b}$, so $x \in \mathfrak{b}$. So $\mathfrak{a}$ is invertible, since each fractional ideal is quotient of integral ideal each fractional ideal is product prime with some valuation in $\mathbb{Z}$ and then also invertible. Then, we prove that if every fractional ideal is invertible then $\mathcal{O}$ is Dedekind.

1. *First let prove that $\mathcal{O}$ is noetherian. For that it is enough to prove that any ideal of $\mathcal{O}$ is finitely generated. In fact, let $\mathfrak{a}$ be an integral ideal since $\mathfrak{a}\mathfrak{a}^{-1} = \mathcal{O}$. In particular, the unit 1 of R can be written as $1 = a_1b_1 + ... + a_nb_n$ with $a_i \in \mathfrak{a}$ and $b_i \in \mathfrak{a}^{-1}$, so that $a = a_1(b_1a) + ... + a_n(b_na) \in (a_1, ..., a_n)$, since $b_ia \in \mathcal{O}$. So that $\mathfrak{a} = (a_1, ..., a_n)$ is finitely generated.*

2. *Now, we prove that any prime ideal is maximal. Let $\mathfrak{p}$ be a prime ideal, and $\mathfrak{m}$ containing $\mathfrak{p}$. As $\mathfrak{m}$ is invertible, there exists an ideal $\mathfrak{a}$ such that $\mathfrak{p} = \mathfrak{m}\mathfrak{a}$. Then, $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{m} \subseteq \mathfrak{p}$. The first case gives $\mathfrak{p} \subseteq \mathfrak{m}\mathfrak{p}$ and by canceling the invertible ideal $\mathfrak{p}$ implies that $\mathfrak{m} = \mathcal{O}$, a contradiction. So the second case must be true and, by maximality of $\mathfrak{m}$, $\mathfrak{p} = \mathfrak{m}$, showing that all prime ideals are maximal.*

3. *Now, let prove that $\mathcal{O}$ is integrally closed. Let x be an element of the field of fraction k of $\mathcal{O}$ and integral over $\mathcal{O}$. Then, we can write $x_n = c_0 + c_1x + ... + c_{n-1}x_{n-1}$ for coefficients $c_k \in \mathcal{O}$. Let $\mathfrak{a}$ be the fractional ideal of $\mathfrak{O}$, $\mathfrak{a} = (1, x, x^2, ..., x^{n-1})$ so that since $x^n \in \mathfrak{a}$, $x\mathfrak{a} \subseteq \mathfrak{a}$. As $\mathfrak{a}$ is invertible, it can be cancelled to give $x \in \mathcal{O}$, showing that R is integrally closed.*